A 3-D Novel Conservative Chaotic System and its Generalized Projective Synchronization via Adaptive Control

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Abstract

This research work proposes a five-term 3-D novel conservative chaotic system with a quadratic nonlinearity and a quartic nonlinearity. The conservative chaotic systems have the important property that they are volume conserving. The Lyapunov exponents of the 3-D novel chaotic system are obtained as $L_1 = 0.0836, L_2 = 0$ and $L_3 = -0.0836$. Since the sum of the Lyapunov exponents is zero, the 3-D novel chaotic system is conservative. Thus, the Kaplan-Yorke dimension of the 3-D novel chaotic system is easily seen as 3.0000. The phase portraits of the novel chaotic system simulated using MATLAB depict the chaotic attractor of the system. This research work also discusses other qualitative properties of the system. Next, an adaptive controller is designed to achieve Generalized Projective Synchronization (GPS) of two identical novel chaotic systems with unknown system parameters. MATLAB simulations are shown to validate and demonstrate the GPS results derived in this work.

Keywords: Chaos, chaotic systems, conservative systems, synchronization, generalized projective synchronization, adaptive control.

1. Introduction

There is great interest in the chaos literature in the discovery of chaotic behaviour in nature and physical systems. Chaotic systems are defined as nonlinear dynamical systems which are very sensitive to initial conditions, topologically mixing and also with dense periodic orbits [1].

A significant development in chaos theory occurred when Lorenz discovered a 3-D chaotic system of a weather model [2]. This was followed by the discovery of many 3-D chaotic systems in the chaos literature such as Rössler system [3], Rabinovich system [4], ACT system [5], Sprott systems [6], Chen system [7], Lü system [8], Shaw system [9], Feeny system [10], Shimizu system [11], Liu-Chen system [12], Cai system [13], Tigan system [14], Colpitt’s oscillator [15], WINDMI system [16], Zhou system [17], etc.

Recently, many 3-D chaotic systems have been discovered such as Li system [18], Elhadj system [19], Pan system [20], Sundarapandian system [21], Yu-Wang system [22], Sundarapandian-Pehlivan system [23], Zhu system [24], Vaidyanathan systems [25-30], Vaidyanathan-Madhavan system [31], Pehlivan-Moroz-Vaidyanathan system [32], Jafari system [33], Pham system [34], etc.

We note that the chaotic systems [2-34] are dissipative systems in which the system limit sets are ultimately confined into a specific limit set of zero volume, and the asymptotic motion of the chaotic system settles onto a strange attractor of the system.

In the chaos literature, there is also an active interest in the discovery of conservative chaotic systems [35], which have the special property that the volume of the flow is conserved.

A classical example of a conservative chaotic system is the Nosé-Hoover system [36, 37], which is modelled by the system of differential equations

\[
\begin{align*}
\frac{dx_1}{dt} &= x_2 \\
\frac{dx_2}{dt} &= -x_1 + x_2x_3 \\
\frac{dx_3}{dt} &= 1 - x_1^2
\end{align*}
\]

The Nosé-Hoover system (1) has the Lyapunov exponents $L_1 = 0.014, L_2 = 0$ and $L_3 = -0.014$. The system (1) is chaotic as it has a positive Lyapunov exponent and is conservative as the sum of the Lyapunov exponents is zero. Thus, the system (1) is volume-conserving.

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Also, the Kaplan-Yorke dimension of the Nosé-Hoover system (1) is easily seen as

\[ D_{KY} = 2 + \sum_{i=1}^{3} \frac{\lambda_i}{|\lambda_i|} = 2 + 1 = 3 \]  

(2)

In this research work, we modify the dynamics of Nosé-Hoover chaotic system (1) and obtain a 3-D novel conservative chaotic system with Lyapunov exponents \( L_1 = 0.0836, \ L_2 = 0, \) and \( \ L_3 = -0.0836. \) Thus, it is clear that the maximal Lyapunov exponent (MLE) of the novel conservative chaotic system is \( L_1 = 0.0836, \) which is greater than the maximal Lyapunov exponent (MLE) of the Nosé-Hoover chaotic system (1). The Kaplan-Yorke dimension of all 3-D conservative chaotic systems is equal to three.

The study of chaos theory has many important applications in science and engineering such as oscillators [38-40], lasers [41-43], robotics [44-47], chemical reactors [48-50], biology [51,52], ecology [53,54], cardiology [55], memristors [56-60], neural networks [61-63], secure communications [64-67], cryptosystems [68-71], economics [72-74], etc.

Chaos control and chaos synchronization are important research problems in the chaos theory. In the last three decades, many mathematical methods have been developed successfully to address these research problems.

The study of control of a chaotic system investigates methods for designing feedback control laws that globally or locally asymptotically stabilize or regulate the outputs of a chaotic system.

Many methods have been developed for the control and tracking of chaotic systems such as active control [75-78], adaptive control [79-85], backstepping control [86-88], sliding mode control [89, 90], etc.

Chaos synchronization problem deals with the synchronization of a couple of systems called the master or drive system and the slave or response system. To solve this problem, control laws are designed so that the output of the slave system tracks the output of the master system asymptotically with time.

Because of the butterfly effect, this is a challenging problem even when the initial conditions of the master and slave systems are nearly identical because of the exponential divergence of the outputs of the two systems in the absence of any control. The synchronization of chaotic systems has applications in secure communications [91-93], cryptosystems [94, 95], encryption [96, 99], etc.

In the chaos literature, many different methodologies have been also proposed for the synchronization and anti-synchronization of chaotic systems such as PC method [100], active control [101-111], time-delayed feedback control [112,113], adaptive control [114-125], sampled-data feedback control [126-129], backstepping control [130-136], sliding mode control [137-143], etc.

Furthermore, we derive an adaptive control law that achieves generalized projective synchronization (GPS) of the identical 3-D novel conservative chaotic systems when the system parameters are unknown. Generalized projective synchronization is a general type of synchronization which generalizes complete synchronization, anti-synchronization, hybrid synchronization, and projective synchronization of chaotic systems. The main synchronization result is proved using adaptive control theory and Lyapunov stability theory. MATLAB simulations are shown in detail to validate and demonstrate the generalized projective synchronization of the identical 3-D novel conservative chaotic systems.

2. A Five-Term 3-D Novel Conservative Chaotic System

The dynamics of the five-term novel 3-D conservative chaotic system is described by

\[
\begin{align*}
\frac{dx_1}{dt} &= x_2 \\
\frac{dx_2}{dt} &= -x_1 + ax_2x_3 \\
\frac{dx_3}{dt} &= b - x_2^4
\end{align*}
\]  

(3)

where \( x_1, x_2, x_3 \) are the states and \( a, b \) are positive parameters.

The nonlinear system (3) depicts a chaotic attractor when the parameter values are taken as:

\[ a = 1, \ b = 1 \]  

(4)

We take the initial conditions as

\[ x_1(0) = 0.2, \ x_2(0) = 0.2, \ x_3(0) = 0.2 \]  

(5)

The 3-D portrait of the strange chaotic attractor (3) for the parameter values (2) and the initial conditions (5) is depicted in Fig. 1, and the 2-D portraits (projections on the three coordinate planes) are depicted in Figs. 2-4.

Fig. 1. The chaotic attractor of the novel chaotic system in \( R^3 \).

Fig. 2. The 2-D projection of the chaotic attractor on the \((x_1, x_2)\) plane.
In vector notation, we may express

\[ \frac{dx}{dt} = f(x) = \begin{bmatrix} f_1(x_1, x_2, x_3) \\ f_2(x_1, x_2, x_3) \\ f_3(x_1, x_2, x_3) \end{bmatrix} \]  

where

\[ \begin{align*}
  f_1(x_1, x_2, x_3) &= x_2 \\
  f_2(x_1, x_2, x_3) &= -x_1 + ax_2x_3 \\
  f_3(x_1, x_2, x_3) &= b - x_2^2 
\end{align*} \]  

We take the parameter values as in the chaotic case, viz. \( a = 1 \) and \( b = 1 \).

Let \( \Omega \) be any region in \( \mathbb{R}^3 \) with a smooth boundary and also \( \Omega(t) = \Phi_t(\Omega) \), where \( \Phi_t \) is the flow of \( f \).

Furthermore, let \( V(t) \) denote the volume of \( \Omega(t) \).

By Liouville’s theorem, we have

\[ \frac{dV}{dt} = \int_\Omega (\nabla \cdot f) \, dx_1 \, dx_2 \, dx_3 \]  

The divergence of the novel chaotic system (3) is easily found as:

\[ \nabla \cdot f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} = 0 + 0 + 0 = 0 \]

Substituting (9) into (8), we obtain the first order ODE

\[ \frac{dV}{dt} = 0 \]  

Integrating (10), we obtain the unique solution as:

\[ V(t) = V(0) \quad \text{for all } t \geq 0 \]

This shows that the 3-D novel chaotic system (3) is volume-conserving. Hence, the system (3) is a conservative chaotic system.

### 3.2. Symmetry and Invariance

It is easy to see that the system (3) is invariant under the coordinates transformation

\[ (x_1, x_2, x_3) \mapsto (-x_1, -x_2, x_3) \]

Thus, the system (3) has rotation symmetry about the \( x_3 \)-axis and any non-trivial trajectory of the system (3) must have a twin trajectory. It is also easy to see that the \( x_3 \)-axis is invariant under the flow of the system (3).

### 3.3. Equilibrium Points

The equilibrium points of the novel chaotic system (3) are obtained by solving the following system of equations (with \( a = 1 \) and \( b = 1 \))

\[ \begin{align*}
  x_2 &= 0 \\
  -x_1 + ax_2x_3 &= 0 \\
  b - x_2^2 &= 0 
\end{align*} \]

Since the first and last equations of the system (13) contradict each other, it is immediate that the system (13) does not admit any solution. Hence, there is no equilibrium for the novel chaotic system (3).

### 3.4. Lyapunov Exponents and Kaplan-Yorke Dimension

For the chosen parameter values (4), the Lyapunov exponents of the novel chaotic system (3) are obtained using MATLAB as:

\[ L_1 = 0.0836, L_2 = 0, L_3 = -0.0836 \]

Since the spectrum of Lyapunov exponents (14) has a positive term \( L_1 \), it follows that the 3-D novel system (1) is chaotic. Since the sum of the Lyapunov exponents is zero, the novel chaotic system is conservative.

The maximal Lyapunov exponent (MLE) of the novel chaotic system (3) is \( L_1 = 0.0836 \).
Also, the Kaplan-Yorke dimension of the novel chaotic system (3) is calculated as:

\[
D_{KY} = 2 + \frac{x_1 + x_2}{|k|} = 2 + 1 = 3 \tag{15}
\]

Fig. 5 depicts the dynamics of the Lyapunov exponents of the novel chaotic system (3).

![Fig. 5. Dynamics of the Lyapunov Exponents of the Novel System.](image)


In this section, new results are derived for an adaptive controller to achieve generalized projective synchronization (GPS) of the identical 3-D novel conservative chaotic systems.

As the master system, we take the novel chaotic system

\[
\begin{aligned}
\frac{dx_1}{dt} &= x_2 \\
\frac{dx_2}{dt} &= -x_1 + ax_2x_3 \\
\frac{dx_3}{dt} &= b - x_4^2
\end{aligned} \tag{16}
\]

where \( x_1, x_2, x_3 \) are state variables and \( a, b \) are unknown, constant, parameters of the system.

As the slave system, we take the novel chaotic system

\[
\begin{aligned}
\frac{dy_1}{dt} &= y_2 + u_1 \\
\frac{dy_2}{dt} &= -y_1 + ay_2y_3 + u_2 \\
\frac{dy_3}{dt} &= b - y_4^2 + u_3
\end{aligned} \tag{17}
\]

where \( y_1, y_2, y_3 \) are state variables and \( u_1, u_2, u_3 \) are adaptive controllers to be designed.

The generalized projection synchronization (GPS) error between the identical chaotic systems is defined as:

\[
\begin{aligned}
e_1(t) &= y_1(t) - \eta_1x_1(t) \\
e_2(t) &= y_2(t) - \eta_2x_2(t) \\
e_3(t) &= y_3(t) - \eta_3x_3(t)
\end{aligned} \tag{18}
\]

In (18), \( \eta_1, \eta_2, \eta_3 \) are real, scaling constants, which are known. (Note that \( \eta_i \) can take both positive and negative values).

The GPS error dynamics is calculated as:

\[
\begin{aligned}
\frac{de_1}{dt} &= y_2 - \eta_1x_2 + u_1 \\
\frac{de_2}{dt} &= -y_1 + \eta_2x_1 + a(y_2y_3 - \eta_2x_2x_3) + u_2 \\
\frac{de_3}{dt} &= b(1 - \eta_3) - y_4^2 + \eta_3x_4^2 + u_3
\end{aligned} \tag{19}
\]

We consider the adaptive control law

\[
\begin{aligned}
u_1 &= -y_2 + \eta_1x_2 - k_1e_1 \\
u_2 &= y_1 - \eta_2x_1 - A(t)(y_2y_3 - \eta_2x_2x_3) - k_2e_2 \\
u_3 &= -B(t)(1 - \eta_3) + y_4^2 - \eta_3x_4^2 - k_3e_3
\end{aligned} \tag{20}
\]

where \( k_1, k_2, k_3 \) are positive gains and \( A(t), B(t) \) are estimates of the unknown parameters \( a, b \) respectively.

The parameter estimation errors are defined by

\[
\begin{aligned}
e_a(t) &= a - A(t) \\
e_b(t) &= b - B(t)
\end{aligned} \tag{21}
\]

Substituting (20) into the error dynamics (19), we get

\[
\begin{aligned}
\frac{de_1}{dt} &= -k_1e_1 \\
\frac{de_2}{dt} &= (a - A(t))(y_2y_3 - \eta_2x_2x_3) - k_2e_2 \\
\frac{de_3}{dt} &= (b - B(t))(1 - \eta_3) - k_3e_3
\end{aligned} \tag{22}
\]

Using (21), we can simplify the error dynamics (22) as:

\[
\begin{aligned}
\frac{de_a}{dt} &= -k_1e_1 \\
\frac{de_b}{dt} &= e_a(y_2y_3 - \eta_2x_2x_3) - k_2e_2 \\
\frac{de_3}{dt} &= e_b(1 - \eta_3) - k_3e_3
\end{aligned} \tag{23}
\]

Differentiating (21) with respect to \( t \), we get

\[
\begin{aligned}
\frac{de_a}{dt} &= -\frac{dA}{dt} \\
\frac{de_b}{dt} &= -\frac{dB}{dt}
\end{aligned} \tag{24}
\]

Next, we use Lyapunov stability theory for finding an update law for the parameter estimates.

Consider the quadratic Lyapunov function defined by

\[
V = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_a^2 + e_b^2), \tag{25}
\]

which is positive definite on \( \mathbb{R}^5 \).

Differentiating \( V \) along the trajectories of (23) and (24), we get

\[
\begin{aligned}
\frac{dv}{dt} &= -k_1e_1^2 - k_2e_2^2 - k_3e_3^2 \\
&+ e_a \left[ e_2(y_2y_3 - \eta_2x_2x_3) - \frac{dA}{dt} \right] \\
&+ e_b \left[ e_3(1 - \eta_3) - \frac{dB}{dt} \right]
\end{aligned} \tag{26}
\]
In view of (26), we take the parameter update law as:

\[
\begin{align*}
\frac{d\hat{A}}{dt} &= e_2(y_2^2y_3^3 - \eta_2z_2x_3^3) \\
\frac{d\hat{B}}{dt} &= e_3(1 - \eta_3)
\end{align*}
\]  

(27)

**Theorem 1.** The adaptive control law (20) and the parameter update law (27) achieve generalized projective synchronization (GPS) between the identical chaotic systems (16) and (17) for all initial conditions, where \(k_1, k_2, k_3\) are positive constants.

**Proof.** We prove this result using Lyapunov stability theory.

For this purpose, we consider the quadratic Lyapunov function \(V\) defined by (25), which is positive definite on \(R^6\).

Substituting the parameter update law (27) into (26), we obtain the time derivative of \(V\) as:

\[
\frac{dV}{dt} = -ke_1e_1^2 - ke_2e_2^2 - ke_3e_3^2,
\]

(28)

which is a negative semi-definite function on \(R^6\).

Thus, we can conclude that the synchronization error \(e(t)\) and the parameter estimation error are globally bounded.

We define \(k = \min\{k_1, k_2, k_3\}\). Then we get

\[
\frac{dV}{dt} \leq -k\|e\|^2 \quad \text{or} \quad k\|e\|^2 \leq -\frac{dV}{dt}
\]

(29)

Integrating the inequality (29) from 0 to \(t\), we get

\[
k \int_0^t \|e(r)\|^2 dr \leq V(0) - V(t)
\]

(30)

From (30), it follows that \(e(t) \in L_2\). Using (23), we can conclude that \(e(t) \to 0\) exponentially as \(t \to \infty\) for all initial conditions \(e(0) \in R^2\). This completes the proof.

For numerical simulations, the parameter values of the novel chaotic systems (16) and (17) are taken as in the chaotic case, viz. \(a = 1\) and \(b = 1\). We take the gain constants as \(k_i = 6\) for \(i = 1, 2, 3\). The GPS constants are taken as \(\eta_1 = 1.2, \eta_2 = -0.7\) and \(\eta_3 = 1.8\).

The initial conditions of the master system (16) are taken as \(x_i(0) = 9.5, y_i(0) = 2.3\) and \(z_i(0) = -6.1\).

The initial conditions of the slave system (17) are taken as \(y_i(0) = 4.8, y_i(0) = -2.9\) and \(y_i(0) = 7.5\).

The initial conditions of the parameter estimates are taken as \(\hat{A}(0) = 4.2\) and \(\hat{B}(0) = 0.2\).

Figs. 6-8 describe the generalized projective synchronization (GPS) of the novel chaotic systems (16) and (17), while Fig. 9 describes the time-history of the synchronization errors \(e_1, e_2, e_3\).
6. Conclusions

In this research work, we have proposed a five-term 3-D novel conservative chaotic system with a quadratic nonlinearity and a quartic nonlinearity. The conservative chaotic systems have the important property that they are volume conserving. Also, the Kaplan-Yorke dimension of any 3-D conservative chaotic system is equal to 3. The Lyapunov exponents of the 3-D novel chaotic system have been obtained as \( L_1 = 0.0836 \), \( L_2 = 0 \) and \( L_3 = -0.0836 \). Also, the maximal Lyapunov exponent of the 3-D novel conservative chaotic system is \( L_1 = 0.0836 \), which is greater than the maximal Lyapunov exponent of the 3-D Nosé-Hoover conservative chaotic system, viz. \( L_1 = 0.014 \). The phase portraits of the novel chaotic system were simulated using MATLAB. We also showed that the 3-D novel conservative chaotic system has no equilibrium points and discussed its symmetry and invariance properties. Next, an adaptive controller was designed to achieve generalized projective synchronization (GPS) of two identical novel chaotic systems with unknown system parameters. Generalized projective synchronization is a general type of synchronization which generalizes common types of synchronization such as complete synchronization (CS), anti-synchronization (AS), hybrid synchronization (HS), projective synchronization (PS), etc. The adaptive GPS synchronization result was established using Lyapunov stability theory. Finally, MATLAB simulations were shown to validate and demonstrate the GPS result derived in this work for identical 3-D novel conservative chaotic systems.

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