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Research Article

Higher Order and Fractional Diffusive Equations

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Abstract

We discuss the solution of various generalized forms of the Heat Equation, by means of different tools ranging from the use of Hermite-Kampé de Fériet polynomials of higher and fractional order to operational techniques. We show that these methods are useful to obtain either numerical or analytical solutions.

Keywords: Heat Equation, Hermite Polynomials, Orthogonal Polynomials.

1 Introduction

The Heat Equation

$$\begin{aligned} &\frac{\partial}{\partial t}F(x,t)=\frac{\partial^2}{\partial t^2}F(x,t)\\ &F(x,0)=g(x) \end{aligned} \tag{1}$$

is one of the most popular equations in Mathematical-Physics [1]. Its solutions are well known and one of the most frequently exploited form is provided by the Gauss transform

$$F(x,t)=\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{+\infty}\exp\left[-\frac{(x-\xi)^2}{4t}\right]g(\xi)\,d\xi \tag{2}$$

which is a kind of convolution of the function $g(x)$ on a Gaussian. This solution holds for positive values of the variable t and when the integral on the r.h.s. of eq. (2) converges.

The Hermite-Kampé de Fériet (H.K.d.F.) polynomials [2]

$$H_n(x,y)=n!\sum_{r=0}^{\left[\frac{n}{2}\right]}\frac{y^r x^{n-2r}}{(n-2r)!r!} \quad , \tag{3}$$

with

$$H_n(x,0)=x^n \quad , \tag{4}$$

are natural solutions of the Heat Equations and indeed they satisfy the recurrences [3, 4]

$$\begin{aligned} &\frac{\partial}{\partial x}H_n(x,y)=nH_{n-1}(x,y)\\ &\frac{\partial}{\partial y}H_n(x,y)=n(n-1)H_{n-2}(x,y) \end{aligned} \tag{5}$$

which can be combined to get

$$\frac{\partial}{\partial y}H_n(x,y)=\frac{\partial^2}{\partial x^2}H_n(x,y) \quad . \tag{6}$$

It is to be stressed that the definition and the previously quoted properties of the $H_n(x,y)$ polynomials are not limited to positive values of y, so that, in principle one can use this family of polynomials as suitable basis to get solutions for eq. (1), which can also be cast in the form [1]

$$F(x,t)=\sum_{n=0}^{+\infty}a_n H_n(x,t) \quad . \tag{7}$$

The validity of the above solution is limited to the fact that $g(x)$ admits the expansion

$$g(x)=\sum_{n=0}^{+\infty}a_n x^n \tag{8}$$

and that the series on the r.h.s. of eq. (7) converges.

Going back to the case having a Gaussian as initial function we note that the solution is compatible with

$$F(x,t)=\frac{1}{\sqrt{1-4t}}\exp\left(-\frac{x^2}{1-4t}\right) \quad , \tag{9}$$

being the negative and positive solutions symmetrical it .is clear that the convergence radius is limited to $|t| < 1/4$ [5].

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In this paper we will consider generalized forms of the equation (1), namely

$$\frac{\partial}{\partial t}F(x,t)=\frac{\partial^{\mu}}{\partial t^{\mu}}F(x,t) \qquad (10)$$
$$F(x,0)=g(x)$$

where μ is any real such that $0<\mu<1$, or is an integer larger than 2.

We will see that a suitable extension of the previous two methods may be an efficient tool to deal with the solution of the evolution equation associated with eq. (10).

2 Higher and fractional order Kampé de Fériet polynomials and solution of the generalized Heat Equation

The higher order HKdF polynomials defined by [2, 6]

$$H_n^{(m)}(x,y)=n!\sum_{r=0}^{\left[\frac{n}{m}\right]}\frac{y^r x^{n-mr}}{(n-mr)!r!} \quad , \qquad (11)$$

are natural solutions of eq. (10) with $\mu=m\in\mathbb{N}$, since they satisfy the recurrences

$$\frac{\partial}{\partial x}H_n^{(m)}(x,y)=nH_n^{(m)}(x,y)$$
$$\frac{\partial}{\partial y}H_n^{(m)}(x,y)=\frac{n!}{(n-m)!}H_{n-m}^{(m)}(x,y) \qquad (12)$$

It is now evident that we can apply the same procedure as before to obtain the solutions of the generalized Heat Equation in the form

$$F(x,t)=\sum_{n=0}^{+\infty}a_nH_n^{(m)}(x,t) \qquad (13)$$

Alternative methods also exploited in the treatment of non local problems in Physics and based on Fourier Transform techniques [7], namely

$$F(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp\left(i^m tk^m\right)\tilde{g}(k)\exp(ikx)\,dk \quad , \qquad (14)$$

where

$$\tilde{g}(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}g(x)\exp(-ikx)\,dx \quad , \qquad (15)$$

is the Fourier Transform associated with the initial function, are of limited usefulness. They are indeed applicable in cases in which the initial function has a Fourier transform and the integral (14) converges.

The fractional order counterparts of (11) are specified by ($0.5\le\mu<1$, $x>0$, $n>0$)

$$H_n^{(\mu)}(x,y)=n!\sum_{r=0}^{\left[\frac{n+\mu}{\mu}\right]}\frac{y^r x^{n-\mu r}}{r!\Gamma(n-\mu r+1)!} \quad , \qquad (16)$$

which strictly speaking are functions and not polynomial functions. The $H_n^{(\mu)}(x,y)$ satisfy all the formal properties of the higher HKdF provided that m be replaced by μ, in particular [2, 8, 9]

$$\frac{\partial}{\partial y}H_n^{(\mu)}(x,y)=\frac{\partial^{\mu}}{\partial x^{\mu}}H_n^{(\mu)}(x,y) \qquad (17)$$
$$H_n^{(\mu)}(x,0)=x^n$$

As a consequence we can use $H_n^{(\mu)}(x,y)$ as a basis to derive the solution of a fractional diffusive equation whose usefulness in applicative problems has been discussed in previous articles, namely

$$F(x,t)=\sum_{n=0}^{+\infty}a_nH_n^{(\mu)}(x,t) \quad . \qquad (18)$$

We have attempted a first benchmark of the correctness of the procedure by checking that the method provides the solution $F(x,t)=g(x+t)$ in the limit $\mu\to1$. This check only cannot be considered sufficient to state the correctness of the procedure and in the forthcoming section we will discuss the comparison with an independent method based on an integral transform technique.

Then, since it is possible to represent the Generalized two-variable Bessel functions in terms of two-variable Hermite polynomials [12], in a forthcoming paper we will investigate how it is possible to obtain different expressions of the above results by using special families of Bessel functions.

3 Conclusions

In [5] has been pointed out that the Laplace-Transform identity

$$\exp\left(-y\sqrt{\delta}\right)=\frac{y}{2\sqrt{\pi}}\int_0^{+\infty}\frac{\exp\left(-\frac{y^2}{4\xi}\right)}{\xi\sqrt{\xi}}\exp(-\xi\delta)\,d\xi \qquad (19)$$

can be exploited to solve equations of the type

$$\frac{\partial}{\partial t}F(x,t)=-\frac{\partial^{1/2}}{\partial x^{1/2}}F(x,t) \qquad (20)$$
$$F(x,0)=g(x)\,,\quad t>0$$

The formal solution of the above equation writes

$$F(x,t)=\exp\left(-t\frac{\partial^{1/2}}{\partial x^{1/2}}\right)g(x) \qquad (21)$$

by using therefore the identity (19), after setting

$$\delta\to\frac{\partial^{1/2}}{\partial t^{1/2}}$$

we find

$$F(x,t)=\frac{t}{2\sqrt{\pi}}\int_0^{+\infty}\frac{\exp\left(-\frac{t^2}{4\xi}\right)}{\xi\sqrt{\xi}}g(x-\xi)\,d\xi \tag{22}$$

In a comparison with the expansion in terms of $H_n^{(\mu)}(x,y)$ functions discussed in the previous section, the two procedures show full agreement and, being independent, we can be sufficiently confident on the reliability of the procedures.

It is worth noting that the transform (22) provides also an independent definition for half order H.K.d.F. polynomials [2, 3], namely ($n \neq 0$)

$$H_n^{(1/2)}(x,-t)=\frac{t}{2\sqrt{\pi}}\int_0^{+\infty}\frac{\exp\left(-\frac{t^2}{4\xi}\right)}{\xi\sqrt{\xi}}(x-\xi)^n\,d\xi \tag{23}$$

which appears the fractional counterpart of a more familiar form

$$H_n(x,t)=\frac{1}{2\sqrt{\pi t}}\int_0^{+\infty}\exp\left[-\frac{(x-\xi)^2}{4t}\right]\xi^n\,d\xi \tag{24}$$

The results we have discussed so far show that an appropriate combination of different techniques may provide a fairly useful tool to deal with the solution of evolution equations of higher order or fractional diffusive type.

Before concluding the paper, we want to stress that the proposed methods are general enough to go beyond the just discussed equations.

We consider indeed the case

$$\begin{aligned}&\frac{\partial}{\partial t}F(x,t)=-\left(x\frac{\partial}{\partial x}\right)^{1/2}F(x,t)\\ &F(x,0)=g(x)\end{aligned} \tag{25}$$

The use of the identities (19)

$$\exp\left(\lambda x\frac{\partial}{\partial x}\right)g(x)=g\left[\exp(\lambda)x\right] \tag{26}$$

leads to the solution

$$F(x,t)=\frac{t}{2\sqrt{\pi}}\int_0^{+\infty}\frac{\exp\left(-\frac{t^2}{4\xi}\right)}{\xi\sqrt{\xi}}g\left[\exp(-\xi)x\right]d\xi \tag{27}$$

The final example we will discuss is the half-order Fokker-Planck equation [10]

$$\frac{\partial}{\partial t}F(x,t)=-\left(x+\frac{\partial}{\partial x}\right)^{\frac{1}{2}}F(x,t) \tag{28}$$

whose solution can be cast in the form

$$F(x,t)=\frac{t}{2\sqrt{\pi}}\int_0^{+\infty}\frac{\exp\left(-\frac{t^2}{4\xi}\right)}{\xi\sqrt{\xi}}\exp\left[-\xi\left(x+\frac{\partial}{\partial x}\right)\right]g(x)\,d\xi \tag{29}$$

In dealing with eq. (29) we have the extra complication that the exponential should be decoupled using an appropriate relation. The use of the Weyl decoupling theorem [3, 4, 11]

$$\exp\left(\hat{a}+\hat{b}\right)=\exp\left(\hat{a}\right)\exp\left(\hat{b}\right)\exp\left(-\frac{k}{2}\right)\quad , \tag{30}$$

where k is a c-number commuting with both the operators $\hat{a}$, $\hat{b}$, yields [1]

$$F(x,t)=\frac{t}{2\sqrt{\pi}}\int_0^{+\infty}\frac{\exp\left[-\frac{t^2}{4\xi}-\xi(x-\xi)\right]}{\xi\sqrt{\xi}}g(x-\xi)\,d\xi \tag{31}$$

Analogous results can be obtained by exploiting a slightly modified version of the expansion method, presented in the previous sections, for a preliminary discussion on this aspect of the problem.

References

1. X. Xu, *Algebraic Approaches to Partial Differential Equations*, Springer Berlin Heidelberg, 2013.
2. C. Cesarano, G. Cennamo, L. Placidi, "Humbert polynomials and functions in terms of Hermite polynomials towards applications to wave propagation", *WSEAS Transactions on Mathematics*, vol. 13, pp. 595–602, (2014).
3. G. Dattoli, C. Cesarano, "On a new family of Hermite polynomials associated to parabolic cylinder functions", *Applied Mathematics and Computation*, vol. **141** (1), pp. 143–149, (2003).
4. S. Khan, A. Al-Gonah, "Multi-variable Hermite matrix polynomials: Properties and applications", *Journal of Mathematical Analysis and Applications*, vol. **412** (1), pp. 222–235, (2014).
5. G. Dattoli, S. Lorenzutta, G. Maino, A. Torre, C. Cesarano, "Generalized Hermite polynomials and supergaussian forms", *Journal of Mathematical Analysis and Applications*, vol. **203** (3), pp. 597–609, (1996).
6. C. Cesarano, "Hermite polynomials and some generalizations on the heat equations", *International Journal Of Systems Applications, Engineering & Development*, vol. **8**, pp. 193–197, (2014).
7. M. Ruzhansky, V. Turunen (Eds.), *Fourier Analysis*, Springer Berlin Heidelberg, (2014).
8. G. Dattoli, S. Lorenzutta, C. Cesarano, "Generalized polynomials and new families of generating functions", *Annali dell'Università di Ferrara*, vol. **47** (1), pp. 57–61, (2001).
9. G. Dattoli, P. Ricci, C. Cesarano, "A note on multi-index polynomials of Dickson type and their applications in quantum optics", *Journal of Computational and Applied Mathematics*, vol. **145** (2), pp. 417–424, (2002).
10. F. N. Bernard Helffer, "Hypoelliptic Estimates and Spectral Theory for Fokker-Planck Operators and Witten Laplacians", *Lecture Notes in Mathematics*, Springer Berlin Heidelberg, (2005).
11. G. Dattoli, S. Lorenzutta, P. Ricci, C. Cesarano, "On a family of hybrid polynomials", *Integral Transforms and Special Functions*, vol. **15** pp. 485–490, (2004).
12. C. Cesarano, D. Assante, "A note on Generalized Bessel Functions", *International Journal of Mathematical Models and Methods in Applied Sciences*, vol. 7(6), pp. 625-629, 2013.