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Lecture Note

Time of flight and range of the motion of a projectile in a constant gravitational field under the influence of a retarding force proportional to the velocity

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Abstract

In this paper we study the classical problem of the motion of a projectile in a constant gravitational field under the influence of a retarding force proportional to the velocity. Specifically, we express the time of flight, the time of fall and the range of the motion as a function of the constant of resistance per unit mass of the projectile. We also prove that the time of fall is greater than the time of rise with the exception of the case of zero constant of resistance where we have equality. Finally we prove a formula from which we can compute the constant of resistance per unit mass of the projectile from time of flight and range of the motion when the acceleration due to gravity and the initial velocity of the projectile are known.

Keywords: Time, flight, coefficient.

1. Introduction

Let us consider the motion of a projectile in a resisting medium under the influence of a constant force of gravity. The medium resistance will give rise to a retarding force. In general, real retarding forces are complicated functions of velocity but in many cases it is sufficient to consider that the retarding force is simply proportional to the velocity. For example if the medium is the air, it is found experimentally, for a relatively small object moving in air with velocities less than about 24m/s that the retarding force is approximately proportional to the velocity [1]. The proportionality case was first examined by Newton in his Principia (1687). Let us call the constant of proportionality constant of resistance. Although the equations of motion can be integrated directly, the dependence of time of flight from the constant of resistance per unit mass of the projectile is given by a transcendental equation for which we have approximate solutions for small values of the independent variable [1]. In this paper we shall try to develop an exact method of expressing time of flight and range as functions of the constant of resistance per unit mass of the projectile for all its possible values.

2. Statement of the problem and solution

A projectile of mass m is fired in a homogeneous medium with an initial velocity $\boldsymbol{\upsilon_0}$ and in a direction making an angle $\hat{\theta}$ with the horizontal ($0<\hat{\theta}\leq\pi/2$). The projectile moves in a constant gravitational field of strength $\boldsymbol{g}$ under the influence of a retarding force proportional to the velocity. Let σ denote the constant of resistance per unit mass of the projectile ($\sigma\geq 0$).

2.1 For $0<\hat{\theta}\leq\pi/2$, let us express the time of flight of the projectile as a function of σ.

Solution

We analyze the motion into horizontal and vertical components with corresponding axes X and Ψ. The projectile is fired from the point of origin of the X-Ψ coordinate system.

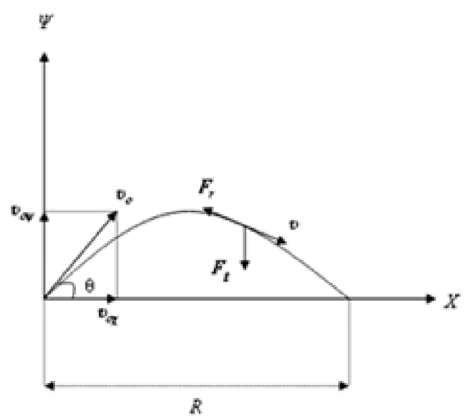

Figure 1. The motion into horizontal and vertical components with corresponding axes X and Ψ

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Let $\chi(t)$, $\psi(t)$ be the coordinates of the projectile as functions of time.

The initial conditions may be stated as

$$\left.\begin{aligned}\chi(0)&=0, & \chi'(0)&=v_0\cos\hat{\theta}\equiv v_{0\chi},\\ \psi(0)&=0, & \psi'(0)&=v_0\sin\hat{\theta}\equiv v_{0\psi},\end{aligned}\right\}\tag{1}$$

where $v_0=|\boldsymbol{v}_0|$, $\quad 0<\hat{\theta}\le\pi/2$.

The projectile moves under the influence of the gravitational force

$$\boldsymbol{F}_g=m\,\boldsymbol{g}\tag{2}$$

and the retarding force

$$\boldsymbol{F}_r=-m\sigma\boldsymbol{v}.\tag{3}$$

From Newton's Second Law the equations of motion are

$$m\chi''(t)=-m\sigma\chi'(t)\tag{4}$$

$$m\psi''(t)=-mg-m\sigma\psi'(t).\tag{5}$$

The solutions of the above Linear Differential Equations are

$$\left.\begin{aligned}\chi(t)&=v_{0\chi}t, & \sigma&=0,\\ \chi(t)&=\frac{v_{0\chi}}{\sigma}\left(1-e^{-\sigma t}\right), & \sigma&\in(0,+\infty)\end{aligned}\right\}\tag{6}$$

and

$$\left.\begin{aligned}\psi(t)&=v_{0\psi}t-\frac{1}{2}gt^2, & \sigma&=0,\\ \psi(t)&=-\frac{g}{\sigma}t+\frac{\left(g+\sigma v_{0\psi}\right)}{\sigma^2}\left(1-e^{-\sigma t}\right), & \sigma&\in(0,+\infty).\end{aligned}\right\}\tag{7}$$

We intend to write time of flight as a function of σ; let $T(\sigma):[0,+\infty)$ be the corresponding function. The time of flight may be found by noticing that $\psi=0$ at the end of the trajectory, hence

$$\psi(T(\sigma))=0\tag{8}$$

By combining Eq.7 and Eq.8 we find

$$\left.\begin{aligned}T(\sigma)&=2a, & \sigma&=0,\\ 1+\sigma a&=\frac{\sigma T(\sigma)}{1-e^{-\sigma T(\sigma)}}, & \sigma&\in(0,+\infty)\end{aligned}\right\}\tag{9}$$

where

$$a\equiv v_{0\psi}g^{-1}\tag{10}$$

From the second equation of Eqs.9, with the help of Lemma 3.1, we find

$$1+\sigma a=\frac{-\sigma T(\sigma)}{e^{-\sigma T(\sigma)}-1}\Leftrightarrow$$

$$1+\sigma a=b(-\sigma T(\sigma))\Leftrightarrow$$

$$T(\sigma)=-\sigma^{-1}b^{-1}(1+\sigma a),\quad \sigma\in(0,+\infty).\tag{11}$$

Briefly we have

$$T(\sigma)=\begin{cases}2a, & \sigma=0\\ -\sigma^{-1}b^{-1}\left(1+\sigma a\right), & \sigma\in(0,+\infty).\end{cases}\tag{12}$$

Lemma 3.2 enables us to show

$$\mathrm{T}(\sigma)=ah(1+\sigma a),\quad \sigma\in[0,+\infty)\tag{13}$$

and then to derive that $T(\sigma)$ is a continuous and strictly monotonic decreasing function over its domain of definition and that its range is the interval $(\alpha,2\alpha]$.

In Fig.2 we have chosen $(1+\sigma\alpha)$ as independent variable and $T(\sigma)/\alpha$ as dependent in order to show the properties of the curve without assigning any value to parameter α.

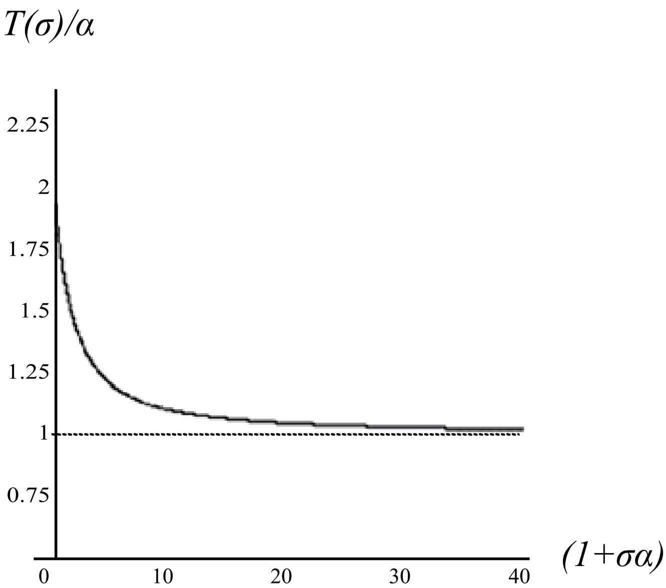

Figure 2. The variation of $T(\sigma)/\alpha$ with $(1+\sigma\alpha)$

In &2.2 we shall seek a way in order to comprehend deeper the limiting behaviour of the curve in Fig.2.

2.2 For $0<\hat{\theta}\le\pi/2$, let us express the time of fall of the projectile as a function of σ and let us prove that it is greater than the time of rise with the exception of the case of zero constant of resistance where we have equality.

Solution

We intend to write time of rise as a function of σ; let $T_f(\sigma): [0,+\infty)$ be the corresponding function.

By solving Eq.5 for $\psi'(t)$ we find

$$\begin{cases} \psi'(t) = v_{0_\psi} - gt, & \sigma=0 \\ \psi'(t) = \dfrac{-g}{\sigma} + \left(v_{0_\psi} + \dfrac{g}{\sigma} \right) e^{-\sigma t}, & \sigma \in (0,+\infty). \end{cases} \tag{14}$$

The time of rise may be found by noticing that $\psi' = 0$ at the highest point of the trajectory, hence

$$\psi'(T_r(\sigma)) = 0. \tag{15}$$

By solving the last equation for T_r we find

$$T_r(\sigma) = \begin{cases} a, & \sigma=0 \\ \sigma^{-1} \ln(1 + a\sigma), & \sigma \in (0,+\infty). \end{cases} \tag{16}$$

From Eq.16, we may derive that $T_r(\sigma)$ is a continuous and strictly monotonic decreasing function over its domain of definition and that its range is the interval

$$\left(\lim_{\sigma \to +\infty} T_r(\sigma), T_r(0) \right] = (0, a].$$

We intend to obtain the time of fall as a function of σ; let $T_f(\sigma): [0,+\infty)$ be the corresponding function. Then we have

$$T_f(\sigma) = T(\sigma) - T_r(\sigma), \qquad \sigma \in [0,+\infty). \tag{17}$$

By combining the last equation with Eq.12 and Eq.16 we find

$$T_f(\sigma) = \begin{cases} a, & \sigma=0 \\ -\sigma^{-1} \left(b^{-1}(1 + a\sigma) + \ln(1 + a\sigma) \right), & \sigma \in (0,+\infty). \end{cases} \tag{18}$$

Let us now try to prove the inequality

$$\begin{cases} T_r(\sigma) \le T_f(\sigma), & \sigma \in [0,+\infty), \\ \text{where the equality holds only for } \sigma = 0. \end{cases} \tag{19}$$

By combining Eq.16 and Eq.18 we find

$$T_r(0) = T_f(0) = a, \tag{20}$$

$$\begin{aligned} T_r(\sigma) - T_f(\sigma) &= \sigma^{-1} \left(2\ln(1+a\sigma) + b^{-1}(1+a\sigma) \right) \\ &= \sigma^{-1} \ln\left((1+a\sigma)^2 e^{b^{-1}(1+a\sigma)} \right) < 0, \quad \sigma \in (0, +\infty), \end{aligned} \tag{21}$$

where, for the determination of the sign of the logarithm in Eq.21 we use Lemma 3.3 by putting $\chi = b^{-1}(1+a\sigma)$ and proving the inequality

$$(1 + a\sigma)^2 e^{b^{-1}(1+a\sigma)} < 1, \qquad \sigma \in (0,+\infty). \tag{22}$$

The proof of inequality (19) is now complete.

Finally the variation of $T_f(\sigma)/\alpha$ with $(1+\sigma\alpha)$ is shown in Fig.3 and Fig.4.

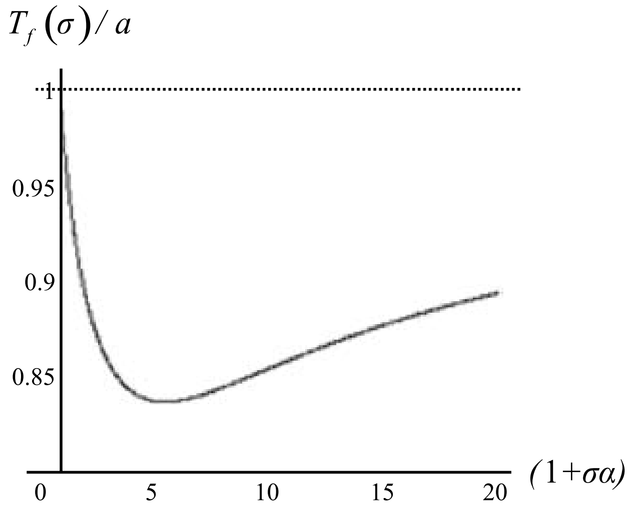

Figure 3. The variation of $T_f(\sigma)/\alpha$ with $(1+\sigma\alpha)$

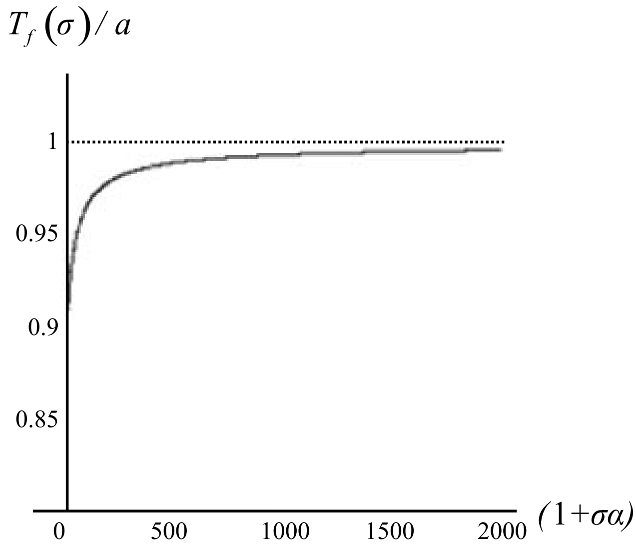

Figure 4. The variation of $T_f(\sigma)/\alpha$ with $(1+\sigma\alpha)$

We notice from Fig.3 and Fig.4 that, as $(1+\sigma\alpha)$ increases, the function $T_f(\sigma)/\alpha$ initially decreases reaching the absolute minimum ($absolute - min\ (T_f(\sigma)/\alpha) \to 0.838274$ when $(1+\sigma\alpha) \to 5.54791$) and then increases approaching the limiting value 1.

For a deeper comprehending of the limiting behaviour of the curve in Fig.4 we will try to develop an approximate treatment of the fall of the projectile, in the vertical sense, for $\sigma ? \alpha^{-1}$.

From Eq.13 and Lemma 3.2 we find

$$\frac{T(\sigma)}{a} \approx 1, \qquad \sigma ? \; a^{-1}. \tag{23}$$

From Eq.16 we obtain

$$\frac{T_r(\sigma)}{a} = \; 1, \qquad \sigma ? \; a^{-1}. \tag{24}$$

Then,

$$\begin{aligned} &\frac{T_f(\sigma)}{a} = \frac{T(\sigma)}{a} - \frac{T_r(\sigma)}{a} \Rightarrow \\ &\frac{T_f(\sigma)}{a} \approx 1, \qquad \sigma ? \; a^{-1}. \end{aligned} \tag{25}$$

By combining Eq.7 and Eq.16 we obtain

$$\psi_{max}(\sigma) = \frac{g}{\sigma} a \left(1 - \frac{T_r(\sigma)}{a}\right), \; \sigma \in (0, +\infty), \tag{26}$$

where $\psi_{max}(\sigma) \equiv \psi(T_r(\sigma)) : [0, +\infty)$ stands for the maximum height reached by the projectile.
From Eqs.(24), (25), (26) we finally get

$$\psi_{max}(\sigma) \approx \upsilon_{ter}(\sigma) T_f(\sigma), \quad \sigma ? \; a^{-1}, \tag{27}$$

where

$$\upsilon_{ter}(\sigma) \equiv \frac{g}{\sigma} : (0, +\infty) \tag{28}$$

stands for the magnitude of the terminal velocity of the projectile [1].

Eq.27 permits us to approximate the motion of the projectile during its fall for $\sigma ? \; \alpha^{-1}$, in the vertical sense, as uniform with speed equal to u_{ter} and may be comprehended as follows.

During projectile's fall, because of the large value of σ compared to α^{-1}, the vertical component of the retarding force becomes so big that it almost cancels the force of gravity in a negligible amount of time in comparison with the time of fall; consequently the projectile approaches its terminal velocity approximately from the highest point of its trajectory.

The above approximate treatment of the fall of the projectile also helps us to comprehend in a deeper way the limiting behaviour of the curve in Fig.2, since $T_r(\sigma ? \; \alpha^{-1})$ is negligible compared to $T_f(\sigma ? \; \alpha^{-1})$.

2.3 For $0 < \hat{\theta} \le \pi / 2$, let us express the range of the projectile as a function of σ.

Solution
Let $R(\sigma) : [0, +\infty)$ be the corresponding function, then the following relation holds

$$R(\sigma) = \chi(T(\sigma)). \tag{29}$$

By putting in Eq.6 $t = T(\sigma)$ and expressing $T(\sigma)$ through Eq.12 we find

$$R(\sigma) = \begin{cases} 2\upsilon_{0\chi} a, & \sigma = 0 \\ \upsilon_{0\chi} \sigma^{-1} \left(1 - e^{b^{-1}(1+\sigma a)}\right), & \sigma \in (0, +\infty). \end{cases} \tag{30}$$

We notice that, for the function b^{-1} of Lemma 3.1, it holds,

$$\begin{aligned} &1 + \sigma a = b\left(b^{-1}(1 + \sigma a)\right) \Leftrightarrow \\ &1 + \sigma a = \frac{b^{-1}(1 + \sigma a)}{e^{b^{-1}(1+\sigma a)} - 1} \Leftrightarrow \end{aligned}$$

$$1 - e^{b^{-1}(1+\sigma a)} = -\frac{b^{-1}(1 + \sigma a)}{1 + \sigma a}, \; \sigma \in (0, +\infty). \tag{31}$$

By combining Eq.30 and Eq.31 we find, for the range,

$$R(\sigma) = \begin{cases} 2\upsilon_{0\chi} a, & \sigma = 0 \\ -\upsilon_{0\chi} \sigma^{-1} \dfrac{b^{-1}(1 + \sigma a)}{1 + \sigma a}, & \sigma \in (0, +\infty), \end{cases} \tag{32}$$

which yields the relation we seek, in its simplest form.

By combining Eq.12 with Eq.32 we find the important relation,

$$R(\sigma) = \upsilon_{0\chi} \frac{T(\sigma)}{1 + \sigma a}, \qquad \sigma \in [0, +\infty) \tag{33}$$

From Eq.33, we may derive that $R(\sigma)$ is a continuous and strictly monotonic decreasing function over its domain of definition and that its range is the interval $\left(\lim_{\sigma \to +\infty} R(\sigma), R(0)\right] = \left(0, 2\upsilon_{0\chi} a\right]$.

Finally the variation of $R(\sigma) / \alpha\upsilon_{o\chi}$ with $(1+\sigma\alpha)$, for $0<\hat{\theta}<\pi/2$, is shown in Fig.5.

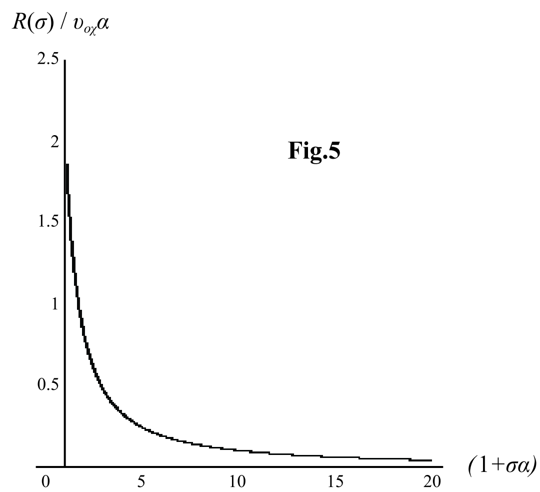

Figure 5. The variation of $R(\sigma) / \alpha\upsilon_{o\chi}$ with $(1+\sigma\alpha)$, for $0<\hat{\theta}<\pi/2$

For the limiting behaviour of the curve in Fig.5, by combining Eq.23 and Eq.33, we obtain the approximate expression

$$\frac{R(\sigma)}{\upsilon_{0\chi}a} \approx \frac{a}{a(1+a\sigma)} \approx \frac{1}{a\sigma},\ \sigma ?\ \ a^{-1},\ 0<\hat{\theta}<\pi/2 \qquad (34)$$

Finally, we shall try to develop an approximate treatment of the trajectory of the projectile, for $\sigma\ ?\ \alpha^{-1}$.

Let $L(\sigma) \equiv \chi(T_r(\sigma)) : [0,+\infty)$ be the displacement of the projectile, along the horizontal axis, during its rise. By combining Eq.6 and Eq.16 we get

$$L(\sigma) = \frac{\upsilon_{0\chi}a}{1+a\sigma},\ \ \sigma \in [0,+\infty) \qquad (35)$$

Let $D(\sigma) \equiv R(\sigma) - L(\sigma) : [0,+\infty)$ be the displacement of the projectile, along the horizontal axis, during its fall. By combining Eq.33 and Eq.35 we obtain

$$\frac{D(\sigma)}{R(\sigma)} = 1 - \frac{\upsilon_{0\chi}a/(1+a\sigma)}{\upsilon_{0\chi}T(\sigma)/(1+a\sigma)} = \\ = 1 - \frac{a}{T(\sigma)},\ \sigma \in [0,+\infty),\ 0<\hat{\theta}<\pi/2 \qquad (36)$$

Finally from Eq.23 and Eq.36 we get

$$\frac{D(\sigma)}{R(\sigma)} \approx 0,\ \ \sigma ?\ \ a^{-1}, 0<\hat{\theta}<\pi/2. \qquad (37)$$

The above approximate relationship permits us to consider that for $\sigma\ ?\ \alpha^{-1}$ the trajectory of the projectile during its fall approximates closely to the vertical line that passes from the highest point reached by it.

By combining both results concerning the fall, the one above and the one in &2.2, we conclude that the motion of the projectile during its fall, for $\sigma\ ?\ \alpha^{-1}$, can be regarded with good approximation, as uniform along the vertical line which passes from the highest point reached by it, with speed equal to its terminal velocity.

From Eqs.(10), (24), (26), (35) we obtain

$$\frac{\psi_{max}(\sigma)}{L(\sigma)} \approx \frac{(g/\sigma)a}{\upsilon_{0\chi}/\sigma} = \\ = \frac{\upsilon_{0\psi}}{\upsilon_{0\chi}} = tan\ \hat{\theta},\ \ \sigma ?\ \ \ a^{-1}, 0<\hat{\theta}<\pi/2 \qquad (38)$$

The above relationship permits us to consider that the trajectory of the projectile during its rise, for $\sigma\ ?\ \alpha^{-1}$, approximates closely to the line along witch it was launched.

2.4 For $0<\hat{\theta}<\pi/2$, let us express the constant of resistance per unit mass of the projectile as a function of the time of flight and range of the motion.

Solution

From Eq.33, by solving for σ we find

$$\sigma = a^{-1}\left(\upsilon_{0\chi}\frac{T(\sigma)}{R(\sigma)} - 1\right), \sigma \in [0,+\infty), 0<\hat{\theta}<\pi/2 \qquad (39)$$

Obviously Eq.39 gives the relation we seek.

3. Appendix

3.1 Lemma

The inverse function for

$$b(\chi) = \begin{cases} \dfrac{\chi}{e^{\chi}-1}, & \chi \in (-\infty,0) \\ 1, & \chi=0 \end{cases} : (-\infty,0] \rightarrow [1,+\infty) \qquad (40)$$

Is

$$b^{-1}(\omega) = -\omega + \sum_{r=1}^{\infty}\frac{r^{r-1}}{r!}\left(\omega e^{-\omega}\right)^r : [1,+\infty) \rightarrow (-\infty,0] \qquad (41)$$

(For the proof of Lemma 3.1 see Ref.[2].)

3.2 Lemma

The function

$$h(\omega) = \begin{cases} \dfrac{b^{-1}(\omega)}{1-\omega}, & \omega \in (1,+\infty) \\ 2, & \omega = 1, \end{cases} \qquad (42)$$

(where $b^{-1}(\omega)$ is the function whose definition has been given in Lemma 3.1),
is a continuous and strictly monotonic decreasing function over its domain of definition and its range is the interval $(1,2]$.

Proof

By using elementary calculus we may derive that $h(\omega)$ is a continuous function on its domain of definition. In order to determine the sign of the first derivative of $h(\omega)$ we consider the composite function

$$(h \circ b)(\chi) \equiv h(b(\chi)) = \begin{cases} \dfrac{\chi}{1-b(\chi)}, & \chi \in (-\infty,0), \\ 2, & \chi=0, \end{cases} \qquad (43)$$

(where $b(\chi)$ is the function whose definition has been given in Lemma 3.1) and then calculate its first derivative

$$(h \circ b)'(\chi) = \frac{1 - b(\chi) + \chi b'(\chi)}{(1 - b(\chi))^2} > 0, \quad \chi \in (-\infty, 0). \tag{44}$$

For the first derivative of $b(\chi)$ it holds

$$b'(\chi) = \frac{e^{\chi} - \chi e^{\chi} - 1}{(e^{\chi} - 1)^2} < 0, \ \chi \in (-\infty, 0). \tag{45}$$

For the first derivative of the composite function $(h \circ b)(\chi)$ it holds

$$(h \circ b)'(\chi) = h'(\omega) b'(\chi), \ \omega = b(\chi), \ \chi \in (-\infty, 0) \tag{46}$$

By combining the last equation with Eq.44 and Eq.45 we find

$$h'(\omega) < 0, \qquad \omega \in (1, +\infty). \tag{47}$$

Hence $h(\omega)$ is a continuous and strictly monotonic decreasing function over its domain of definition and its range is the interval $\left(\lim_{\omega \to +\infty} h(\omega), h(1)\right] = (1, 2]$

3.3 Lemma

For the function $b(\chi)$ of Lemma 3.1 it holds

$$e^{\chi} b^2(\chi) < 1 \ \text{ where } \chi \in (-\infty, 0). \tag{48}$$

Proof

We notice

$$\begin{gathered} e^{\chi} b^2(\chi) < 1 \Leftrightarrow e^{\frac{\chi}{2}} b(\chi) < 1 \Leftrightarrow \\ 0 < 1 - e^{\chi} + \chi e^{\frac{\chi}{2}}, \quad \chi \in (-\infty, 0) \end{gathered} \tag{49}$$

where the last inequality can be easily proved by using elementary calculus.

4. Conclusions

In this paper we examined the motion of a projectile in a constant gravitational field under the influence of a retarding force proportional to the velocity and developed an exact method to express time of flight, time of fall and range of the projectile, as functions of the constant of resistance per unit mass of the projectile for all its possible values. Then we expressed the constant of resistance per unit mass of the projectile as a function of the time of flight and range of the motion. Mathematically the above method is based on the inversion of the function

$$b(\chi) = \begin{cases} \dfrac{\chi}{e^{\chi} - 1}, & \chi \in (-\infty, 0) \\ 1, & \chi = 0 \end{cases} : (-\infty, 0] \to [1, +\infty).$$

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