Journal of Engineering Science and Technology Review 2 (1) (2009) 118-122

JOURNAL OF
Engineering Science and Technology Review

www.jestr.org

Research Article

Thin film flow of non-Newtonian fluids on a vertical moving belt using Homotopy Analysis Method

H. Nemati, M. Ghanbarpour*, M. Hajibabayi, Hemmatnezhad

University of Mazandaran, Department of Mechanical Engineering P. O., Box 484, Babol, Iran.

Received 14 December 2008; Revised 14 July 2009; Accepted 31 August 2009

Abstract

In this study, the problem of thin film flow of two non-Newtonian fluids namely, a Sisko fluid and an Oldroyd 6-constant fluid on a vertical moving belt that modeled by a system of nonlinear differential equations is studied. The system is solved using the Homotopy Analysis Method (HAM), which yields an analytic solution in the form of a rapidly convergent infinite series with easily computable terms. Homotopy analysis method contains the auxiliary parameter $\hbar$, which provides us with a simple way to adjust and control the convergence region of solution series. By suitable choice of the auxiliary parameter $\hbar$, reasonable solutions for large modulus can be obtained.

Keywords: thin film, non-Newtonian fluids, HAM.

Introduction

Recently, finding analytical approximating solutions of nonlinear equations has widespread applications in numerical mathematics and applied mathematics and there has appeared an ever increasing interest of scientists and engineers in analytical techniques for studying nonlinear problems. Homotopy Analysis Method has been proposed by Liao and a systematic and clear exposition on this method is given in [1-2]. HAM is a powerful mathematical technique and has already been applied to several nonlinear problems [3-21]. This method contains an auxiliary parameter h which provides us with a simple way to adjust and control the convergence region of the solutions. The equations modeling non-Newtonian incompressible fluid flow give rise to highly nonlinear differential equations. Such non-Newtonian fluids find wide applications in commerce, industry and have now become the focus of extensive study.

In this study, HAM is applied to problem of thin film flow of two non-Newtonian fluids namely, a Sisko fluid [22] and an Oldroyd 6-constant fluid on a vertical moving belt. Using HAM, we have obtained meaningful solutions for v which clearly shows the capability of this method for solving equations with high nonlinearities and controlling the convergence region of solutions. The HAM solutions for two considered examples in comparison with HPM solutions of [23] admit a remarkable accuracy.

1. Basic ideas of homotopy analysis method

To describe the basic ideas of the HAM, consider the following differential equation,

$$\mathcal{N}[y(t)]=0, \tag{1}$$

where $\mathcal{N}$ is a nonlinear operator, t denotes the independent variable, and $y(t)$ is an unknown function respectively. By means of generalizing the traditional homotopy method, we have the so-called zeroth-order deformation equation as

$$(1-q)\mathcal{L}\left[\psi(t,q)-y_0(t)\right]=q\,\hbar\,\mathcal{N}\left[\psi(t,q),q\right], \tag{2}$$

where $\psi(t,q)$ is an unknown function, $\mathcal{L}$ is an auxiliary linear operator, $q\in[0,1]$ an embedding parameter, $\hbar$ a non-zero auxiliary parameter, and $y_0(t)$ is an initial guess of $y(t)$. It is important to note that, one has freedom to choose auxiliary parameters such as $\hbar$ and $\mathcal{L}$ in HAM. Obviously, when $q=0$ and $q=1$, both

$$\psi(t;0)=y_0(t) \quad \text{and} \quad \psi(t;1)=y(t). \tag{3}$$

Thus as q increases from 0 to 1, the solution $\psi(t;q)$ varies smoothly from the known initial guess $y_0(t)$ to the solution $y(t)$. By expanding $\psi(t;q)$ into Taylor series of the embedding parameter q and using (3), we have

* E-mail address: morteza.ghanbarpour@gmail.com
ISSN: 1791-2377

$$\psi(t;q) = y_0(t) + \sum_{n=1}^{+\infty} y_n(t)\, q^n, \tag{4}$$

where

$$y_n(t) = \frac{1}{n!} \left. \frac{\partial^n \psi(t;q)}{\partial q^n} \right|_{q=0}, \tag{5}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, are so properly chosen, then the series (4) converges at $q=1$ and

$$y(t) = y_0(t) + \sum_{n=1}^{+\infty} y_n(t) \tag{6}$$

The governing equations of y_n can be deduced from the zeroth-order deformation equations (2) and (3). Define the vector

$$\overrightarrow{y_n} = \{y_0(t), y_1(t), \ldots, y_n(t)\}, \tag{7}$$

Now substituting Eq. (5) into Eq. (2) and differentiating Eq. (2) n times with respect to the embedding parameter q, then dividing by $n!$, and finally setting $q=0$, we have the n[th]-order deformation equation

$$\mathcal{L}[y_n(t) - \chi_n y_{n-1}(t)] = \hbar R_n(y_{n-1}), \tag{8}$$

subjected to boundary conditions

$$y_n(0) = 0, \quad y'_n(0) = 0, \tag{9}$$

where prime denotes the derivative with respect to t and

$$R_n(y_{n-1}) = \frac{1}{(n-1)!} \left. \frac{\partial^{n-1} \mathcal{N}[\psi(t;q), q]}{\partial q^{n-1}} \right|_{q=0}, \tag{10}$$

and

$$x_k = \begin{cases} 0 & k \le 1 \\ 1 & k > 1 \end{cases} \tag{11}$$

It should be noted that $y_n(t)$ $(n \ge 1)$ is governed by the linear equation (8) together with the linear boundary conditions that come from the original problem.

Problem formulation

We consider a container having a non-Newtonian fluid in it [24-29]. A wide moving belt passes through this container, which moves vertically upward with constant velocity U_0 as shown in Fig.1 Since the belt moves upward and passes through the fluid, it picks up a film fluid of thickness d. Due to gravity, the fluid film tends to drain down the belt. For simplicity, the following assumptions are made:

(i) The flow is in steady state.
(ii) The flow is laminar and uniform.
(iii) The film fluid thickness d is uniform.

We choose an xy-coordinate system and position x-axis parallel to the fluid and normal to the belt, y-axis upward along the belt and z-axis normal to the xy-plane. The only velocity component is in the y-direction, therefore,

$$V = (0, v(x), 0) \tag{12}$$

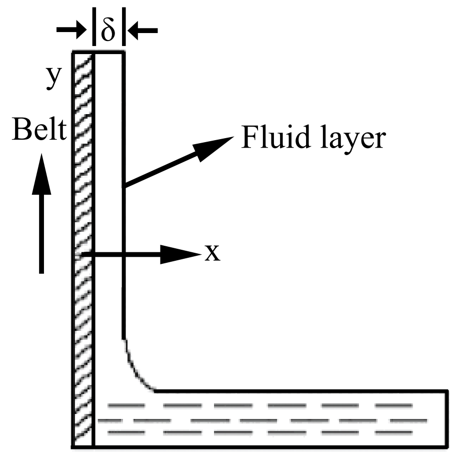

Figure 1. Geometry of the flow of moving belt through a non-Newtonian fluid

and also the extra stress tensor is function of x only, that is

$$S = S(x), \tag{13}$$

Eq. (12) satisfies the continuity Equation identically.

In the sequel, we first derive the flow equations for a Sisko fluid and an Oldroyd 6-constant fluid separately and then solve the resulting equations by using homotopy perturbation method.

Governing equation for a Sisko fluid

Consider the following differential equation of a Sisko fluid over a moving belt [30]

$$\frac{d^2 v}{dx^2} + nb \left[\frac{dv}{dx}\right]^{n-1} \frac{d^2 v}{dx^2} - k = 0 \tag{14}$$

subjected to the boundary conditions

$$\begin{aligned} v &= 1 \qquad x = 0, \\ \frac{dv}{dx} &= 0 \qquad x = 1, \end{aligned} \tag{15}$$

For HAM solutions, we choose the initial guesses and auxiliary linear operator in the following forms:

$$v_0'' = 0, \rightarrow v_0 = \frac{k}{2}(x^2 - 2x) - k, \tag{16}$$

Zeroth –order deformation problems

$$(1-p)L\left[v(x,p) - v_0(x)\right] = p\hbar N\left[v(x,p)\right], \tag{17}$$

$$v(0,p) = 0 \qquad v'(0,p) = 1 \tag{18}$$

$$N[v(x,p)] = \frac{d^2 v(x,p)}{dx^2} + \\ + nb\left[\frac{dv(x,p)}{dx}\right]^{n-1} \frac{d^2 v(x,p)}{dx^2} - k, b = const. \tag{19}$$

For $p=0$ and $p=1$ we have:

$$v(x,0) = v_0(x), \qquad v(x,1) = v(x),$$

When p increases from 0 to 1 and $v(x,p)$ vary from $v_0(x)$ to $v(x)$. Due to Tailor's series with respect to p we have:

$$v(x,p) = v_0(x) + \sum_{m=1}^{\infty} v_m(x) p^m, \\ v_m(x) = \frac{1}{m!} \frac{\partial^m (v(x,p))}{\partial p^m} \tag{20}$$

In which $\hbar$ is chosen in such a way that this series is convergent at ($p=1$).

Therefore we have through Eq. (20) that:

$$v(x) = v_0(x) + \sum_{m=1}^{\infty} v_m(x), \tag{21}$$

Convergence of the HAM solution

As pointed out by Liao, the convergence and rate of approximation for the HAM solution strongly depends on the values of auxiliary parameter $\hbar$. Fig. (2) clearly depict that the ranges, for admissible values of $\hbar$ is -1.4 < $\hbar$ < -0.6. Our calculations clearly indicate that series (21) converge for whole region of x when $\hbar = -1$.

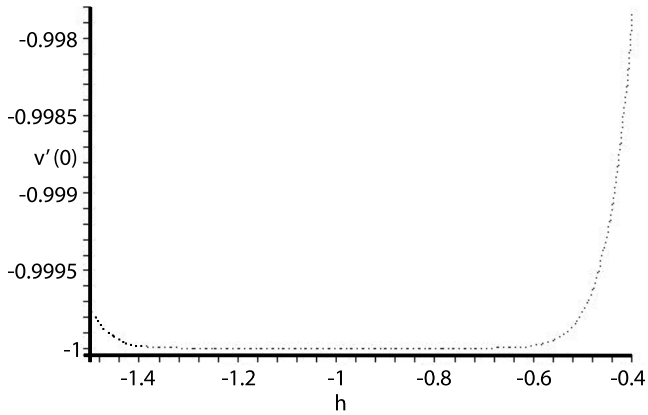

Figure 2. Variation of $\hbar$ for 12[th]-order of approximation

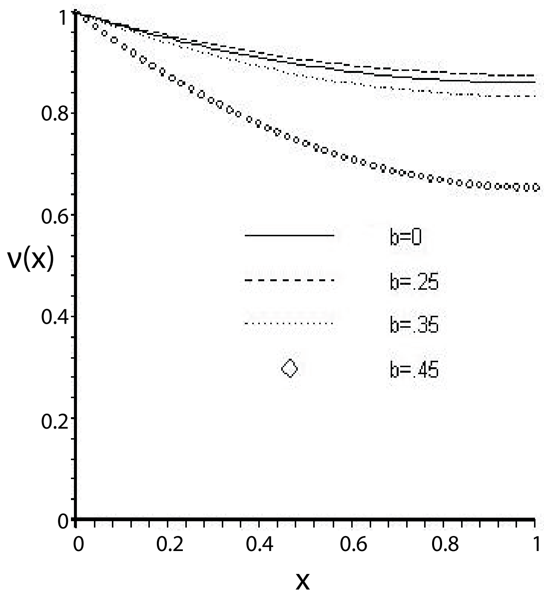

Figure 3. Variation of $v(x)$ for n=1 and k=1

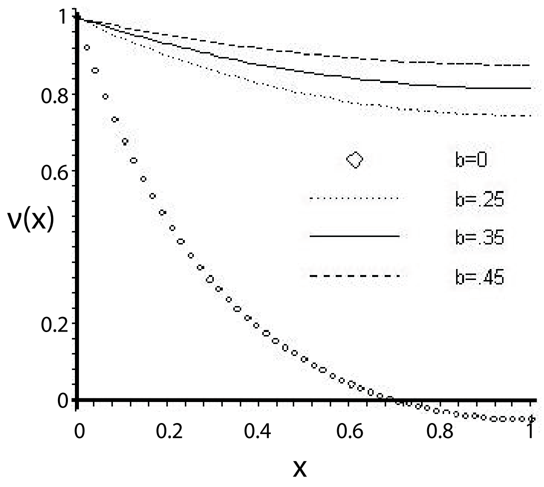

Figure 4. Variation of $v(x)$ for n=2 and k=1

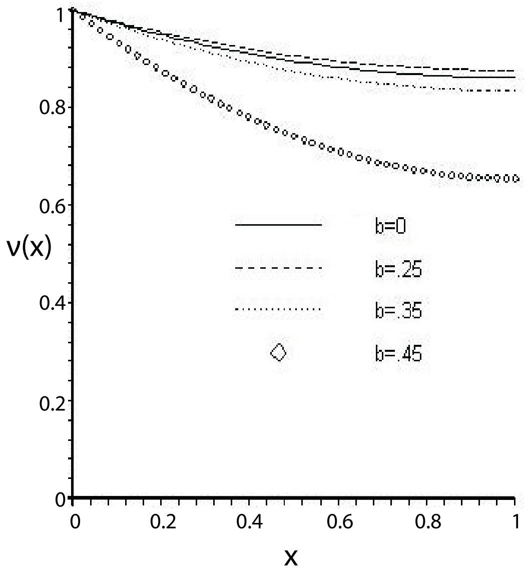

Figure 5. Variation of $v(x)$ for n=3 and k=1

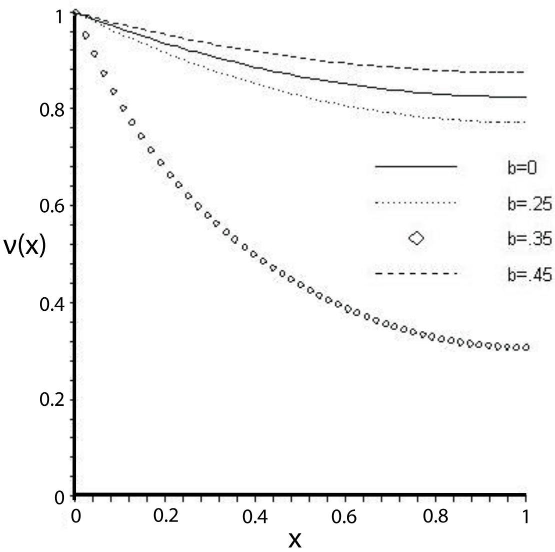

Figure 6. Variation of $v(x)$ for n=4 and k=1

Application of HAM to the problem of Oldroyd 6-constant fluid

Our differential equation with the boundary conditions is [30]

$$\frac{d^2v}{dx^2}+(3a_1-a_2)(\frac{dv}{dx})^2\frac{d^2v}{dx^2}+a_1a_2(\frac{dv}{dx})^4\frac{d^2v}{dx^2}$$
$$-m(1+a_2(\frac{dv}{dx})^2)^2=0, a_1, a_2=const. \tag{22}$$

$$v=1 \qquad x=0,$$
$$\frac{dv}{dx}=0 \qquad x=1, \tag{23}$$

For HAM solutions, we choose the initial guesses and auxiliary linear operator in the following forms:

$$v_0''=0, \rightarrow v_0=\frac{m}{2}(x^2-2x)-m, \tag{24}$$

$$(1-p)L\left[v(x,p)-v_0(x)\right]=p\hbar N\left[v(x,p)\right], \tag{25}$$

$v(0,p) = 0 \qquad v'(0,p) = 1$

$$N[v(x,p)]=\frac{d^2v}{dx^2}+(3a_1-a_2)(\frac{dv}{dx})^2\frac{d^2v}{dx^2}+$$
$$+a_1a_2(\frac{dv}{dx})^4\frac{d^2v}{dx^2}-m(1+a_2(\frac{dv}{dx})^2)^2, \tag{26}$$

For $p=0$ and $p=1$ we have:

When p increases from 0 to 1 $v(x,p)$ vary from $v_0(x)$ to $v(x)$. Due to Tailor's series with respect to p we have:

$$v(x,p)=v_0(x)+\sum_{m=1}^{\infty}v_m(x)p^m,$$
$$v_m(x)=\frac{1}{m!}\frac{\partial^m(v(x,p))}{\partial p^m} \tag{27}$$

In which $\hbar$ is chosen in such a way that this series is convergent at ($p=1$).

Therefore we have through Eq. (27) that:

$$v(x)=v_0(x)+\sum_{m=1}^{\infty}v_m(x), \tag{28}$$

Convergence of the HAM solution

As pointed out by Liao, the convergence and rate of approximation for the HAM solution strongly depends on the values of auxiliary parameter $\hbar$. Fig.(8) clearly depict that the ranges, for ad-

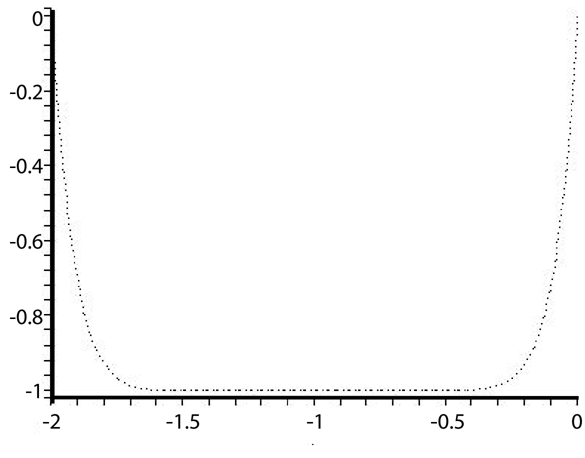

Figure 7. Variation of $\hbar$ for 8th-order of approximation

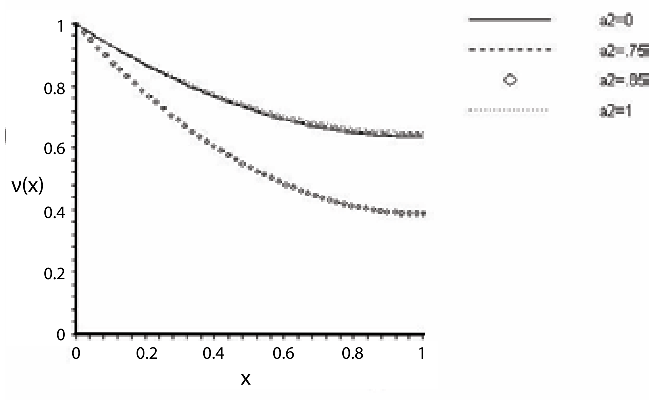

Figure 8. Variation of $v(x)$, for m=1and α_1=0.1

missible values of $\hbar$ is -1.7 < $\hbar$ < -0.3. Our calculations clearly indicate that series (28) converge for whole region of x when $\hbar = -1$.

Conclusions

In this paper, the capability of HAM for obtaining approximate solutions of the velocity profile of thin film flow of non-Newtonian fluids over a moving belt is shown. First HAM is applied to the problem of a Sisko fluid and the problem of an Oldroyd 6-constant fluid is considered for further studies. Using HAM, we have achieved an analytical solution for the velocity of the belt. The results are compared with those of obtained via HPM which clarifies the effectiveness of this method. It is observed from the figures that the speed of the belt decreases as the non-Newtonian effects increase.

References

1. S.J. Liao, The proposed homotopy analysis technique for the solution of nonlinear problems, PhD thesis, Shanghai Jiao Tong University, (1992).
2. S.J. Liao, Beyond Perturbation: Introduction to the Homotopy Analysis Method, Chapman & Hall/CRC Press, Boca Raton, (2003).
3. S.J. Liao, A second-order approximate analytical solution of a simple pendulum by the process analysis method, J. Appl. Mech. 59 (1992) 970–975.
4. S.J. Liao, An explicit totally analytic approximate solution for Blasius viscous flow problems. Int.J. Non-Linear Mech. 34 (1999) 759–769.
5. S.J. Liao, On the homotopy analysis method for nonlinear problems. Appl. Math. Comput. 147 (2004) 499–513.
6. S.J. Liao, A new branch of solutions of boundary-layer flows over an impermeable stretched plate. Int. J. Heat Mass Transfer, 48 (2005) 2529–2539.
7. S.J. Liao, K.F. Cheung, Homotopy analysis of nonlinear progressive waves in deep water, J. Eng. Math. 45 (2) (2003) 105–116.
8. S. Abbasbandy, The application of the homotopy analysis method to nonlinear equations arising in heat transfer, Phy. Lett. A 360 (2006) 109–113.
9. S. Abbasbandy, The application of the homotopy analysis method to solve a generalized Hirota–Satsuma coupled KdV equation, Phy. Lett. A 361 (2007) 478–483.
10. M. Sajid, T. Hayat, The application of homotopy analysis method for thin film flow of a third order fluid, Chaos Solitons Fractal, in press.
11. M. Sajid, T. Hayat, S. Asghar, On the analytic solution of the steady flow of a fourth grade fluid, Phy. Lett. A 355 (2006) 18–24.
12. T. Hayat, Z. Abbas, M. Sajid, Series solution for the upper-convected Maxwell fluid over a porous stretching plate, Phy.Lett. A 358 (2006) 396–403.
13. T. Hayat, M. Khan, S. Asghar, Homotopy analysis of MHD flows of an Oldroyd 8-constant fluid, Acta Mech. 168 (2004) 213–232.
14. T. Hayat, M. Khan, M. Ayub, On the explicit analytic solutions of an Oldroyd 6-constant fluid, Int. J. Eng. Sci. 42 (2004) 123–135.
15. T. Hayat, M. Khan, M. Ayub, Couette and Poiseuille flows of an Oldroyd 6-constant fluid with magnetic field, J. Math. Anal. Appl. 298 (2004) 225–244.
16. T. Hayat, M. Sajid, On analytic solution of thin film flow of a fourth grade fluid down a vertical cylinder, Phy. Lett. A (2007) 316–322.
17. T. Hayat, T. Javed, On analytic solution for generalized three-dimensional MHD flow over a porous stretching sheet, Phy. Lett. A 370 (2007) 243–250.
18. M. Sajid, A.M. Siddiqui, T. Hayat, Wire coating analysis using MHD Oldroyd 8-constant fluid, Int. J. Eng. Sc. 45 (2007) 381–392.
19. T. Hayat, M. Sajid, Homotopy analysis of MHD boundary layer flow of an upper-convected Maxwell fluid, Int. J. Eng. Sc. 45 (2007) 393–401.
20. T. Hayat, N. Ahmed, M. Sajid, S. Asghar, On the MHD flow of a second grade fluid in a porous channel. Com. and Math. App. 54 (2007) 407–414.
21. M. Sajid, T. Hayat, Thin film flow of an Oldroyd 8-constant fluid: An exact solution, Phy. Lett, A 372 (2008) 1827–1830.
22. Sisko AW. The flow of lubrication greases. Int. Eng. Chem. 50 (1958) 1789–1792.
23. Siddiqui AM, Ahmed M, Ghori QK. Thin film flow of non-Newtonian fluids on a moving belt, Chaos, Solitons & Fractals, article in press.
24. T. Hayat, N. Ali, Effect of variable viscosity on the peristaltic transport of a Newtonian fluid in an asymmetric channel, App. Math. Mod. 32 (2008) 761–774.
25. T. Hayat, Z. Abbas, M. Sajid, Heat and mass transfer analysis on the flow of a second grade fluid in the presence of chemical reaction. Phy. Lett. A 372 (2008) 2400–2408.
26. M. Khan, Z. Abbas, T. Hayat, Analytic solution for flow of Sisko fluid through a porous medium, Transp Porous Med. 71 (2008) 23–37.
27. M. Sajid, T. Hayat, Series solution for steady flow of a third grade fluid through porous space, Transp Porous Med. 71 (2008) 173–183.
28. T. Hayat, Z. Abbas, Channel flow of a Maxwell fluid with chemical reaction, Z. Angew. Math. Phy. 59 (2008) 124-144.
29. T. Hayat, Z. Abbas, T. Javed, Mixed convection flow of a micropolar fluid over a non-linearly stretching sheet, Phy. Lett. A 372 (2008) 637–647.
30. S. Asghar, T. Hayat, A. H. Kara, Exact solutions of thin film flows, Nonlinear Dynamics, 50 (2007) 229–233.